COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA **14** (1) (1994) 51–61

A CHARACTERISATION OF SPREADS OVALLY-DERIVED FROM DESARGUESIAN SPREADS

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Received October 31, 1990
Revised February 20, 1991

We characterise all spreads that are obtainable from Desarguesian spreads by replacing a partial spread consisting of translation ovals; the corresponding "ovally-derived" planes are generalised André planes, of order 2^N, although not all generalised André planes are ovally-derived from Desarguesian planes. Our analysis allows us to obtain a complete classification of all nearfield planes that are ovally-derived from Desarguesian planes. It turns out that whether or not a nearfield plane is ovally-derived from a Desarguesian plane depends solely on the parameters q and r, where GF(q) is the kern, and r is the dimension of the plane. Our results also imply that a Hall plane of even order q^2 can be ovally-derived from a Desarguesian spread if and only if q is a square.

1. Introduction

In [5] we considered a natural André type correspondence between affine translation planes that admit translation ovals (relative to the natural axis) and what we called transversal spreads. From the point of view of this article, it is probably better to consider this correspondence in projective rather than vectorial terms, as follows.

Definition. Let S be a spread in $\wp = PG(2N-1,2)$, consisting of 2^N+1 pairwise skew subspaces, each of dimension $N-1$. Then an $N-1$-dimensional subspace τ of $\wp$ is a transversal to S, relative to the carrier-set $\{X,Y\} \subset S$, if

$$|\gamma \cap \tau| = 1 \quad \forall \quad \gamma \in S - \{X,Y\}$$
$$|\gamma \cap \tau| = 0 \quad \text{if} \quad \gamma \in \{X,Y\}.$$

In terms of the above notation, we showed in [5, Theorem 2.3] that the translation plane π_S associated with S admits a translation oval relative to the natural axis (often abbreviated to T-oval) precisely when S admits a transversal τ, and the generic T-oval in the plane π_S is an image of some such τ, under a translation. Moreover, the two points on ℓ_∞, corresponding to the "non-centres" of the T-oval, are just the two slopes associated with the carriers, the definition of a transversal could equally well have been specified without reference to its carriers, as indicated below

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AMS subject classification codes (1991): 51 A 40, 51 E 21

Remark. An $(N-1)$-dimensional subspace τ of $\wp = PG(2N-1,q)$ is a transversal to a spread S in $\wp$ precisely when:

$$(*) \qquad |\gamma \cap \tau| \leq 1 \quad \forall \quad \gamma \in S$$

and (*) cannot hold unless $q=2$.

The purpose of the present article is to consider the possibility of a spread S admitting a covering by pairwise skew transversals in the following sense.

Definition. A transversal cover of a spread S in $\wp = PG(2N-1,2)$, relative to a pair of distinct component $\{X,Y\}$, is a collection Θ of pairwise skew transversals to S such that:

(i) Each transversal in Θ is skew to both X and Y, i.e., all members of Θ share the same carrier-set $\{X,Y\}$;

(ii) $\bigcup\Theta = \bigcup\Sigma$, where $\Sigma = S - \{X,Y\}$

Thus a transversal cover Θ of S, when it exists, is a replacement for the partial spread $\Sigma \subset S$, and we let $\partial_\Theta S$, or just ∂S, denote the corresponding replaced spread $\mathcal{J} = \Theta \cup \{X,Y\}$; we say $\mathcal{J}$ is *ovally derived* from S, relative to the transversal cover Θ. Notice that S is itself ovally-derived from $\mathcal{J}$, i.e.,

$$\partial_\Theta S = \mathcal{J} \Longrightarrow \partial_\Sigma \mathcal{J} = S, \quad \text{where } \bigcup\Theta = \bigcup\Sigma \subset S$$

In general, all spreads ovally-derived from a given spread S, relative to a fixed pair of carriers $\{X,Y\}$, can be ovally-derived from each other.

The question of oval derivation has been considered in a more general context by Assmus and Key [1]. They have also considered a potential technique for carrying out oval derivations of a nearfield spread S, based on searching for a replaceable transversal cover consisting of an orbit of a transversal (i.e., a translation oval) under one of the homology groups of the nearfield spread. They pointed out that this technique, when successful, would yield another nearfield spread $\mathcal{J}$ such that the multiplication table of the two nearfields could be chosen to be isomorphic. In particular, this means that the technique does not allow one to break out of Desarguesian spreads by oval derivation.

One of the goals of the present note, is to demonstrate that the oval derivation of a spread S often leads to a new spread $\mathcal{J}$ whose geometric properties are significantly different from those of the original spread S. Thus we show that many nearfields and generalised André systems can be ovally-derived from a Desarguesian spread, and hence also from each other. In particular, it is easy to arrange for the oval derivation of many nearfield planes to generalised André planes that cannot be coordinated by nearfields.

The above remarks are consequence of the main result of this article, a characterization of all spreads that are ovally-derived from Desarguesian spreads. Our conclusion, which is ultimately underpinned by Payne's classification of the translation ovals in Desarguesian spreads (c.f. Hirschfeld [4, Theorem 8.5.4]), implies that any spread ovally-derived from Desarguesian spreads is a generalised André spread. However, it is far from true that every generalised André spread is ovally-derived from a Desarguesian spread: thus we shall see that for many non-prime integers $N > 1$ there exists non-Desarguesian André planes of order 2^N, π_1 and π_2, such that π_1 is ovally derived from a Desarguesian spread but π_2 cannot be ovally derived from a Desarguesian spread.

Our main result can be stated in terms of what we have chosen to call λ-conics [5]. Roughly speaking, a λ-conic is a natural analogue of the normalised generic translation oval in Desarguesian planes, shown by Payne [4] to be of form "$y = x^{2^M}$". Note that λ-conics are natural generalisations of the Denniston ovals [2]. In [5] we showed that given any $n = 2^N$, $N = sr$, a non-prime, there exists a non-Desarguesian André spread of order $n = 2^N$ that admits λ-conics, and that in certain cases (e.g., whenever N is a prime power) every generalised André spread of order 2^N admits λ-conics. On the other hand, quite often (for instance, whenever N is even but not a power of 2) there exist André planes of order 2^N that do *not* admit λ-conics. It is also readily seen that any λ-conic is part of an oval cover, in any plane containing it. In this article we show:

Theorem 3.2. *A plane is ovally derived from a Desarguesian spread if and only if it is a generalised André spread of order 2^N that admits λ-conics; this plane can be chosen to be non-Desarguesian precisely when N is a composite integer.*

Though, as mentioned above, it is possible to find many generalised André planes that are not ovally-derived from Desarguesian spreads, we have so far been unable to obtain a satisfactory description of precisely which generalised André spreads are derivable from Desarguesian spreads (i.e., admit λ-conics). However, in the present article, we shall give a complete answer to this question for all nearfields spreads; † in particular we shall have established that a nearfield plane is ovally-derived precisely when it arises from a "strong Dickson" pair, an easily recognisable pair of integers.

In the final section we observe that a Hall plane of even order q^2 is ovally-derived from a Desarguesian spread if and only if q is a square. Although the existence of translation ovals in Hall planes of even order has been established by Korchmáros [6], the possibility of ovally deriving the Hall planes has not been demonstrated until now.

2. λ-conics

This section is mainly a survey of some of the facts concerning λ-conics developed in [5]. We begin with some notation that will be used throughout the paper:

Notation. $V = \mathcal{F}_2^N \oplus \mathcal{F}_2^N$, where $\mathcal{F}_2 = GF(2)$. $X = \mathcal{F}_2^N \oplus \mathbf{O}$, $Y = \mathbf{O} \oplus \mathcal{F}_2^N$, and $y = Tx := \{(x, Tx) : x \in \mathcal{F}_2^N\}$, whenever T is an $N \times N$ matrix over $\mathcal{F}_2$.

Now recall that a spread set $\mathcal{M}$ of $N \times N$ matrices over $\mathcal{F}_2$ consists of 2^N matrices such that $\mathbf{O}_N$ and $\mathbf{I}_N \in \mathcal{M}$, and two distinct members in $\mathcal{M}$ differ by a non-singular matrix. The corresponding spread in V will be denoted by:

$$\Delta_{\mathcal{M}} = \{y = Mx : M \in \mathcal{M} - \{\mathbf{O}_N\}\} \cup \{X, Y\}$$

and the affine translation plane, whose lines are the cosets of the components of $\Delta_{\mathcal{M}}$, will be denoted by $\pi_{\mathcal{M}}$. Now it is fairly easy to verify (c.f. [5. Theorem 2.3]

† As we are considering even order spreads, all nearfields may be assumed to be regular, and hence generalised André systems.

that the T-ovals (i.e., translation ovals relative to the translation axis) of $\pi_{\mathcal{M}}$ are precisely the cosets of the transversals to $\Delta_{\mathcal{M}}$; in particular T-ovals with carrier-set $\{X,Y\}$ can be characterised in matrix-theoretic terms using the following

Definition. A *transversal* to a spread set $\mathcal{M}$ consists of a matrix $T \in GL(N,2)$ such that

$$\operatorname{rank}[M-T]=1 \quad \forall \quad M \in \mathcal{M}^* = \mathcal{M} - \{\mathbf{O}_N\}$$

and $\overline{\mathcal{M}} = \mathcal{M} \cup \{T\}$ is the corresponding *transversal* spread set.

It can be verified (see [5, Section 2]) that:

Proposition 1. *The following are equivalent for a spread set* $\mathcal{M}$.

(1) T *is a transversal to the spread set* $\mathcal{M}$;
(2) $y=Tx$ *is a transversal to* $\Delta_{\mathcal{M}}$, *with carrier-set* $\{X,Y\}$;
(3) T *is a non-singular matrix such that* $Mx = Tx$ *has a unique non-zero solution* $x = x_M$, *whenever* $M \in \mathcal{M}^*$.

In the context of this article, we shall be mainly concerned with the case when $\mathcal{M}$ is the natural spread set of a generalised André plane. We shall use the following notation for such systems.

Notation.

(1) $\mathcal{F} = GF(2^N) \supset \mathcal{K} = GF(q) = GF(2^s)$; in particular $N = rs$ where $r = \dim[\mathcal{F}:\mathcal{K}]$.
(2) $\mathcal{Q}_\lambda = (\mathcal{F},+,\circ)$ is the generalised André system with standard addition (inherited from $\mathcal{F}$) and with multiplication $\circ$ defined by a map $\lambda : \mathcal{F}^* \mapsto I_r = \{1,\ldots,r-1\}$, satisfying the Foulser requirements [7, Lemma 10.1], and the condition:

$$m \circ x = \begin{cases} mx^{\left(q^{\lambda(m)}\right)} & m \neq 0 \\ 0 & \text{if } m = 0. \end{cases}$$

(3) The natural spread set associated with $\mathcal{Q}_\lambda$ consists of the set of matrices identified with the linear maps of type $x \mapsto m \circ x$, in $GL(\mathcal{F},+) \cup \{\mathbf{O}_N\}$, and the corresponding spread in $V = \mathcal{F} \oplus \mathcal{F}$ (where we identify $\mathcal{F}$ with $\mathcal{F}_2^N$) is denoted by Δ_λ; in particular X and Y are in the spread Δ_λ. The translation plane associated with this spread will be denoted by π_λ.

The main theme of our study in [5] considers the possibility of sets of type $y = x^{2^M}$ $(=\{(x,x^{2^M}) : x \in \mathcal{F}\} \subset V = \mathcal{F} \oplus \mathcal{F})$ being T-ovals in the generalised André spread Δ_λ. In this context, proposition 1 above translates to the following criterion:

Proposition 2. $y = x^{2^M}$ *is a transversal to the generalised André spread* Δ_λ *(with carrier-set* $\{X,Y\}$*) iff*

$$(s\lambda(f) - M, N) = 1 \quad \forall \quad f \in \mathcal{F}^*.$$

The translation ovals (i.e., transversals) of Δ_λ which are expressible in the form "$y = x^{2^M}$" will henceforth be referred to as **λ-conics**. The following corollary is an immediate consequence of the above propositions.

Corollary 3. *The following are equivalent:*

(1) $y = fx^{2^M}$ *is a* T*-oval in* π_λ *(i.e., a transversal of the associated spread* Δ_λ *coordinatised by* $\mathcal{Q}_\lambda$*);*

(2) $y = x^{2^M}$ *is a* λ*-conic;*

(3) $\{y = fx^{2^M} : f \in \mathcal{F}^*\}$ *is an oval cover for* Δ_λ *relative to the natural carrier-set* $\{X, Y\}$.

We shall call oval covers of type (3), viz. $\{y = fx^{2^M} : f \in \mathcal{F}^*\}$, **$\Lambda$-covers**, and translation ovals of type $y = fx^{2^M}$ will be called **Λ-conics**; thus λ-conics are normalised versions of Λ-conics. Note that a translation oval which is a λ-conic may cease to be a λ-conic, if the plane is recoordinatised by another generalised André system with a different unit point, although it continues to be a Λ-conic. This follows from the fact that the Λ-conic are precisely the translation ovals that extend to Λ-covers, and that Λ-covers turn out to be geometric invariants (c.f., Theorem 3.2).

Using the characterization of λ-conics given a proposition 2 we showed in [5] that at least one non-Desarguesian André plane of even order $n = 2^N$ admits a λ-conic, provided of course that a (non-Desarguesian) generalised André plane of order n exists.

Result 4. If $n = 2^N > 8$, and N is not a prime integer, then there is a (non-Desarguesian) affine André plane π, of order n, such that π admits a T-oval, viz., a λ-conic.

On the other hand, we also showed that very often generalised (non-Desarguesian) André planes of the same order coexist such that at least one of them cannot admit Λ-conics. For instance, we established

Result 5. If $n = 2^N > 4$ is a square integer then, provided N is not a power of 2, there are André planes π_1 and π_2, both of order n such that π_1 admits a λ-conic (as stated in the previous theorem) but π_2 does not.

We show now that the generalised André spreads admitting λ-conics are precisely the class of spreads that are ovally-derived from Desarguesian spreads. In section 4 we determine precisely which nearfields spreads are ovally-derived from Desarguesian spreads, essentially by refining the above two results.

3. The geometry of λ-conics

We start by formulating Payne's classification of the translation ovals that exists in Desarguesian spreads, in a form convenient for our purposes. Thus we need to consider transversals (i.e., translation ovals) in the Desarguesian spread $\Delta_{\mathcal{F}}$ on $V = \mathcal{F} \oplus \mathcal{F}$, with component set given by

$$\Delta_{\mathcal{F}} = \{\text{“}y = mx\text{”} : m \in \mathcal{F}\} \cup \{\mathbf{O} \oplus \mathcal{F}\}.$$

Since Aut$\Delta_{\mathcal{F}}$ is 2-transitive on the components of $\Delta_{\mathcal{F}}$, we restrict ourselves to transversals with carrier-set $\{X, Y\}$, specified as usual by $X := \mathcal{F} \oplus \mathbf{O}$ and $Y := \mathbf{O} \oplus \mathcal{F}$. Thus we see transversals of type $y = Ax$, where A is an additive bijection of $(\mathcal{F},+)$ such that $mx=Ax$ has a unique non-zero solution $x=x_m$ in $\mathcal{F}$, for every $m\in\mathcal{F}^*$ (c.f. Proposition 2.1(3)). By Vaughan's theorem

$$A\mathbf{x} = \sum_{i=0}^{N-1} \alpha_i x^{2^i}, \quad \alpha_i \in GF(2^N)$$

and so $\mathbf{y}=A\mathbf{x}$ is a transversal precisely when

$$m = \sum_{i=0}^{N-1} \alpha_i x^{2^i-1}$$

has a unique solution $x=x_m$ in $\mathcal{F}^*$ for every non-zero $m\in\mathcal{F}^*$, or equivalently when

$$(*) \qquad f(x) = \sum_{i=0}^{N-1} \alpha_i x^{2^i-1}$$

induces a bijection of $\mathcal{F}^*$.

But since the degree of $f(x)\leq 2^{N-1}-1<2^N-2$, $f(x)$ must also be a bijection of $\mathcal{F}=GF(2^N)$ itself, mapping 0 to 0 (e.g., see [4, 1.3(iv), p2]).

This can only occur when $f(x)=\alpha_k x^{2^k-1}$ for some $k>0$, see [4, Lemma 8.5.3, p.184]. Thus we have the following version of Payne's theorem.

Result 1. (Payne, c.f. Hirschfeld [4, Theorem 8.5.4]). Every transversal of the Desarguesian spread $\Delta_{\mathcal{F}}$ is of type $y = fx^\sigma$, where $f \in \mathcal{F}^* = GF(2^N)^*$ and σ ($\neq$ identity) is an element of Aut$\mathcal{F}$.

Now consider an oval cover of $\Delta_{\mathcal{F}}$, and again without loss of generality assume the carriers of the covers are X and Y. If $y=fx^\sigma$ and $y=fx^\tau$ are in the cover then $\sigma=\tau$, for otherwise the cover would have two distinct transversals overlapping, at $(1,f)$. Hence by result 1 above any cover of $\Delta_{\mathcal{F}}$ is associated with a function λ: $\mathcal{F}^*\mapsto \text{Aut}\mathcal{F}\backslash\{\text{ identity }\}$ such that the cover is of type

$$\Theta_\lambda = \{y = fx^{\lambda(f)} \mid f \in \mathcal{F}^*\}$$

So Δ_λ, the spread obtained by oval derivation of the Desarguesian spread $\Delta_{\mathcal{F}}$ by Θ_λ under spread on $V=\mathcal{F}\oplus\mathcal{F}$ given by $\Delta_\lambda=\{X,Y\}\cup\Theta_\lambda$. Now consider the image of Δ_λ under the following mapping

$$\Lambda : V \mapsto V$$
$$(x,y) \mapsto \left(x, y^{\lambda(1)^{-1}}\right).$$

As $\Lambda \in GL(V,+)$, maps Δ_λ to an isomorphic spread $A_\Lambda \supseteq \{X,Y\}$, such that $A_\Lambda\backslash\{X,Y\}$ consists of all components of type

$$y = f^{\lambda(1)^{-1}} x^{\lambda(f)(\lambda(1))^{-1}} \ \forall\ f \in \mathcal{F}^*.$$

Now A_Λ is clearly a generalised André plane, based on $\mu(f)=[\lambda(\lambda(a)f)](\lambda(a))^{-1}$, and

$$f \circ x = fx^{\mu(f)}$$

(with $1 \circ x = 1x^1 = x$). Moreover the line $y = fx$, of Desarguesian spread $\Delta_{\mathscr{F}}$ gets shifted under Λ to the oval $\Lambda(y = fx)$, of A_Λ, given by

$$\begin{aligned} \Lambda(\{(x, fx) : x \in \mathscr{F}\}) &= \{(x, (fx)^\theta) : x \in \mathscr{F}\} \text{ where } \theta = (\lambda(1))^{-1} \\ &= \{(x, f^\theta x^\theta) : x \in \mathscr{F}\} \\ &= \text{“} y = f^\theta x^\theta \text{”, whenever } f \neq 0. \end{aligned}$$

Thus A_Λ admits $\Lambda(\Delta_{\mathscr{F}} \backslash \{X, Y\})$ as an oval cover $\Theta_\Lambda = \{y = fx^\theta : f \in \mathscr{F}^*\}$, since f^θ ranges over $\mathscr{F}^*$.

Notice that Θ_Λ is a Λ-cover of A_Λ. Thus we have used Payne's theorem establish the first part of the following

Theorem 2. (a) *Let* (Δ, Θ) *be a Desarguesian spread with an oval cover. Then the corresponding derived cover* $(\partial\Delta, \partial\Theta)$ *is isomorphic with a* Λ*-cover of a generalised André plane.*

(b) *If* (A, Ψ) *is a* Λ*-cover of a generalised André plane then it is isomorphic with* $(\partial\Delta, \partial\Theta)$*, for some Desarguesian spread* (Δ, Θ).

Proof of (b). If $\mathscr{Q}_\lambda = (\mathscr{F}, +, \circ)$ is a generalised André system based on $\lambda : \mathscr{F} \mapsto I_r$, and such that π_λ admits the Λ-cover

$$\Theta_\lambda = \{y = fx^{2^M} : f \in \mathscr{F}^*\}$$

then $(\partial\Pi_\lambda, \partial\Theta_\lambda)$ is such that $\partial\Pi_\lambda = \{X, Y\} \cup \Theta_\lambda$ and this spread is easily seen to be Desarguesian by noting that it is the image of $\Delta_{\mathscr{F}}$ under the map

$$\begin{aligned} \Lambda : \mathscr{F} \oplus \mathscr{F} &\to \mathscr{F} \oplus \mathscr{F} \\ (x, y) &\mapsto (x, y^\alpha) \end{aligned}$$

where

$$\begin{aligned} \alpha : \mathscr{F} &\to \mathscr{F} \\ x &\mapsto x^{2^M}. \end{aligned}$$

Hence (b) follows.

Corollary 3. *The spreads ovally-derived from Desarguesian spreads are precisely the generalised André planes that admit a* Λ*-cover.*

We stress that not all generalised André planes admit canonical covers. Moreover, those that do, may also admit other types of covers.

4. Ovally-derived nearfield planes

We now classify the ovally-derived nearfield spreads.

Lemma 1. *Let $\mathscr{N}$ be nearfield of order 2^N and dimension r over its kern $GF(q)=GF(2^s)$. Then $\mathscr{N}$ may be expressed as a generalised André system of type $\mathscr{Q}_\lambda=(\mathscr{F},+,\circ)$ where λ satisfies the additional requirement that $\lambda(\mathscr{F}^*)=I_r$.*

Proof. Being of even order, $\mathscr{N}$ must be a Dickson nearfield [7, p.35], and hence a generalised André system, r-dimensional over its kern $GF(q)$. Now $\lambda(\mathscr{F}^*)=I_r$ is implied, for instance, by Foulser [3, Example 3.2, p.383].

Lemma 2. *Let $\mathscr{I}_N$ and $\mathscr{I}_N^*$ denote respectively the ring of integers modulo N, the set of invertible elements in $\mathscr{I}_N$. If $z\in\mathscr{I}_N$ is nilpotent (i.e. there exists a positive integer n with $z^n=0$) and $M\in\mathscr{I}_N^*$ then $M-z\in\mathscr{I}_N^*$.*

Proof. Consider the following elementary identity with $x=M$ and $y=z$

$$x^k-y^k=(x-y)\sum_{i=0}^{k-1}x^iy^{(k-1)-i}.$$

Now choosing n so that $z^n=0$, and $k\geq n$ such that the Euler phi-function $\phi(N)\mid k$, the above identity implies that $(x-y)$ is invertible in $\mathscr{I}_N$, as required.

Corollary 3. $(s\lambda(\mathscr{F}^*)-M,N)=1$ *whenever $M\in\mathscr{I}_N^*$ and s is any nilpotent element in $\mathscr{I}_N$.*

Lemma 4. *If $\lambda(\mathscr{F}^*)=I_r$ and s is not nilpotent in $\mathscr{I}_N$, then to each $M\in\mathscr{I}_N^*$ corresponds an $f_M\in\mathscr{F}^*$ such that $(s\lambda(f_M-M,N)>1$.*

Proof. $\exists p$ (prime) dividing N such that $(p,s)=1$, since s is not nilpotent in $\mathscr{I}_N$. Hence $\exists a_M,b_M\in\mathscr{I}$ such that

$$\begin{aligned}sa_M+b_Mp=M&\Rightarrow sa_M-M=-b_Mp\\&\Rightarrow s(ar+R_M)-M=-b_Mp\exists R_M\in I_r \text{ (dividing } a_M \text{ by } r)\\&\Rightarrow sra+sR_M-M=-b_Mp\\&\Rightarrow sR_M-M=bp \text{ since } p\mid N=sr\\&\Rightarrow (sR_M-M,N)\geq p\exists R_M\in I_r\\&\Rightarrow (s\lambda(f_M)-M,N)>1, \text{ as required.}\end{aligned}$$

Thus Corollary 3 and Lemma 4, when combined with Proposition 2.2 yield

Theorem 5. *Suppose $\lambda(\mathscr{F}^*)=I_r$ defines a generalised André plane π_λ of order $2^N=2^{sr}$ (and it's easy to get such André systems if $2^s-1\geq r$). Then the following are equivalent*
(1) *s is nilpotent in $\mathscr{I}_N$, i.e., all primes dividing N divide s.*
(2) *$y=x^{2^M}$ is a λ-conic for some $M\in\mathscr{I}_N^*$.*

(3) *$y=x^{2^M}$ is a λ-conic for all $M\in\mathcal{Z}_N^*$.*

The following terminology will prove convenient in discussing the content of the above and related results.

Definition. The plane π_λ, associated with the generalised André system $\mathcal{Q}_\lambda = (\mathcal{F}^*,+,\circ)$, is considered saturated with λ-conic if $y=x^{2^M}$ is a λ-conic whenever $M\in\mathcal{Z}_N^*$.

Thus π_λ is saturated with λ-conics if and only if every potential λ-conic, satisfying the necessary condition $M\in Z_N^*$, turns out to be a λ-conic. Hence the theorem above implies that if $\lambda(\mathcal{F}^*)=I_r$ then π_λ is saturated with λ-conics if it admits even one. It is also worth noting that since collineations map all the Λ-conics through a point to another such collection, the notion of π_λ being saturated does not depend on the generalised André system used to coordinatise it.

Specialising to nearfields, Lemma 1 when combined with the above theorem implies the following

Corollary 6. *An even order nearfield plane of order q^r with kern $GF(q)$ is devoid of λ-conics or saturated with λ-conics, the latter iff the prime divisors of r divide $\log_2 g$.*

A nearfield of order 2^{sr}, and with kern $GF(2^s)$, exists precisely when every prime p dividing r also divides 2^s-1 (c.f. Foulser [3, Examples 3.2]). Thus, by Corollary 6 the existence of λ-conics, in r-dimensional nearfields of order q^r, is equivalent to $2^s\equiv 1 \pmod p$ and $s\equiv 0 \pmod p$, for all primes p dividing r. We therefore introduce the following

Definition. $(q=2^s,r)$ is a strong Dickson pair iff all the prime divisors of r divide s and 2^s-1.

Now Corollary 6 may be restated in the following terms.

Corollary 7. *An even order nearfield plane of order q^r, with kern $GF(q)$ and dimension r, is either devoid of λ-conics or is saturated with λ-conics, the latter iff $(q=2^s,r)$ is a strong Dickson pair.*

The generic strong Dickson pairs arising from any given r can be explicitly determined as follows. Since r is odd (because its prime divisors divide 2^s-1), it has prime power decomposition of type $r=p_1^{\alpha_1}\dots p_k^{\alpha_k}$, where each of the primes p_i are odd. Thus $(2^s,r)$ is a strong Dickson pair precisely when every p_i divides s and also 2^s-1. The latter condition is equivalent to $ord_{p_i}(2)\,|\,(s)$ where $ord_p(2)$ denotes the multiplicative order of 2 (mod p), for odd p. Hence $(2^s,r)$ is a strong Dickson pair precisely when s is divisible by $\prod_{i=1}^k p_i$ and by LCM $\{ord_{p_i}(2):k\geq i\geq 1\}$. But this just means that s is divisible by the LCM of $\{p_1,\dots,p_k\}\cup\{ord_{p_i}(2):k\geq i\geq 1\}$. Hence Corollary 7 may be restated as follows.

Theorem 8. *Let $(q=2^s,r)$ be a pair of integers, where r is odd, with prime factor decomposition $r=p_1^{\alpha_1}\dots p_k^{\alpha_k}$, and s is any multiple of the LCM of $\{p_1,\dots,p_k\}\cup\{ord_{p_i}(2):k\geq i\geq 1\}$. Then nearfields of order q^r, with kern $GF(q)$, do exist and the corresponding nearfield spreads always admit λ-conics (and are saturated with them). Conversely the only nearfield spreads that admit λ-conics are those just described.*

The geometric interpretation of the above result is that we can recognise the nearfield spreads that are ovally-derived from Desarguesian spreads just by inspecting their parameters (q and r).

Theorem 9. *A nearfield spread with kern $GF(q)$, $q=2^s$, and dimension r is ovally-derived from a Desarguesian spread if and only if $(2^s,r)$ is a strong dickson pair, explicitly: s is any multiple of the $\{p_1,\ldots,p_k\}\cup\{ord_{p_i}(2): k\geq i\geq 1\}$, where r is any odd integer whose distinct prime factors are $p_1,\ldots,p_k$.*

Ovally-derived Hall planes

Corollary 3.3 enables us to decide which Hall planes are ovally-derived from Desarguesian planes, if we observe that Hall planes, and more generally 2-dimensional André planes of order q^2, admit λ-conics precisely when $q = 2^s$ is a square. All one needs to do is to observe that the corresponding André spreads admit λ-conics precisely when q is a square, and this follows easily from Proposition 2.2 (c.f. [5, Corollary 3.4]). Thus we have

Proposition 1. *Let π be an André plane of order q^2 with kern $GF(q)$; in particular π may be any Hall plane of order q^2. Then π is ovally-derived from a Desarguesian spread precisely when q is a square.*

The above proposition throws some light on the Korchmáros translation ovals [6] that exist in the Hall planes of even order q^2, where $q=2^s$. The proposition implies that, at least when q is non-square, the translation ovals of Korchmáros cannot be part of an oval-cover that derives into a Desarguesian spread. In particular, the Hall planes, with non-square kerns seem to be the only known of generalised André planes that admit translation ovals but not λ-conics. It would be interesting to have a larger stock of such examples.

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